



IIT ROORKEE



NPTEL ONLINE
CERTIFICATION COURSE

QUANTITATIVE INVESTMENT MANAGEMENT

LECTURE 48 Stochastic Calculus

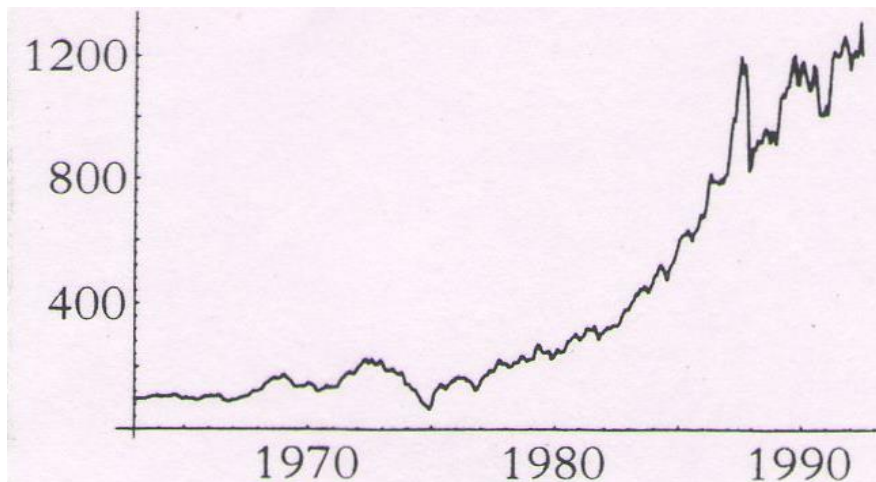
J P Singh

Department of Management Studies

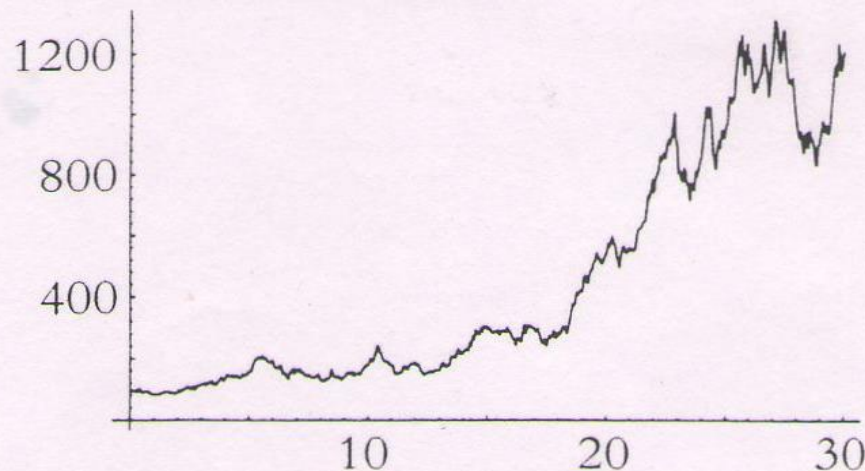


EXPONENTIAL BM OR GEOMETRIC BM

$$X_t = X_0 \exp(\mu t + \sigma W(t))$$



UK FTA index, 1963–92



Exponential Brownian motion

FURTHER GENERALIZATION: THE ITO PROCESS

- The infinitesimal increment of an Ito process can be expressed as:
- $dx = a(x,t)dt + b(x,t)dW$



STOCHASTIC CALCULUS



For the Newtonian derivative, we have :

$$f'(x) = \lim_{dx \rightarrow 0} \frac{f(x + dx) - f(x)}{dx} \quad \text{or}$$

$$\lim_{dx \rightarrow 0} f(x + dx) = f(x) + f'(x)dx \quad (1)$$

If, we expand $f(x + dx)$ as a Taylor series to second order

$$f(x + dx) = f(x) + f'(x)dx + \frac{1}{2}f''(x)(dx)^2 \quad (2)$$

For an arbitrary curve $y = f(x)$, (2) coincides with (1) if dx is so small that $(dx)^2$ can be neglected.

Indeed, that is the relevance behind setting $\lim_{dx \rightarrow 0}$

- In the limiting case, $dx \rightarrow 0$, thus, the points $P(x,y)$ and $Q(x+dx,y+dy)$ on the curve $y=f(x)$ are assumed to be so close that the curve $y=f(x)$ joining them may be approximated by a straight line (with slope dy/dx) and therefore, we may approximate:
- $dy=(dy/dx)dx$ where (dy/dx) is the slope of **STRAIGHT LINE PQ.**
- Thus, at the infinitesimal level, **we assume that the curve is constructed by an assortment of infinitesimal straight lines.**

To understand, what happens in case of a BM increment dW_t instead of dx , we do a Taylor expansion in both cases : For $y = f(x)$, we have,

$$dy = df(x) = f'(x)dx + \frac{1}{2}f''(x)(dx)^2 + \dots$$

$$df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)dW_t \cdot dW_t + \dots$$

$$dW_t \cdot dW_t = Z^2 dt \text{ so } E(Z^2) = 1, \text{Var}(Z^2) = 2$$

$E(dW_t \cdot dW_t) = dt$, $\text{Var}(dW_t \cdot dW_t) = 2(dt)^2$
so that $dW_t \cdot dW_t$ has value dt & is not stochastic.
Hence, it cannot be IGNORED.

BROWNIAN MOTION

- Just as deterministic curves can be considered as comprising of infinitesimal increments of straight lines, stochastic trajectories may be assumed to be formed by infinitesimal increments of Brownian motion.

MEAN AND VARIANCE OF SQ OF STANDARD NORMAL VARIATE

- $E(Z^2)=1$
- $VAR (Z^2)=2$



$$E(Z^4) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^4 e^{-\frac{1}{2}z^2} dz = 2 \frac{1}{\sqrt{2\pi}} \int_0^{\infty} z^4 e^{-\frac{1}{2}z^2} dz \quad (\text{Even function})$$

Let $\frac{1}{2}z^2 = t$ so that $zdz = dt$

$$E(Z^4) = 2 \frac{1}{\sqrt{2\pi}} \int_0^{\infty} (2t)^{3/2} e^{-t} dt = \frac{4}{\sqrt{\pi}} \int_0^{\infty} t^{3/2} e^{-t} dt = \frac{4}{\sqrt{\pi}} \Gamma(5/2).$$

Now, $\Gamma(n+1) = n\Gamma(n)$; $\Gamma(1/2) = \sqrt{\pi}$ so that

$$E(Z^4) = \frac{4}{\sqrt{\pi}} \frac{3}{2} \Gamma(3/2) = \frac{4}{\sqrt{\pi}} \frac{3}{2} \frac{1}{2} \Gamma(1/2) = \frac{4}{\sqrt{\pi}} \frac{3}{2} \frac{1}{2} \sqrt{\pi} = 3;$$

$$E(Z^2) = 1; \text{Var}(Z^2) = E(Z^4) - [E(Z^2)]^2 = 3 - 1 = 2$$

$$\Gamma(1/2) = \int_0^{\infty} t^{-1/2} e^{-t} dt \text{ Let } t = u^2, dt = 2u du$$

$$\Gamma(1/2) = \int_0^{\infty} t^{-1/2} e^{-t} dt = 2 \int_0^{\infty} e^{-u^2} du = \int_{-\infty}^{\infty} e^{-u^2} du$$

$$\text{Let } u = \frac{v}{\sqrt{2}}, du = \frac{dv}{\sqrt{2}} \text{ so that}$$

$$\Gamma(1/2) = \int_{-\infty}^{\infty} e^{-u^2} du = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}v^2} dv = \frac{1}{\sqrt{2}} \sqrt{2\pi} = \sqrt{\pi}$$

$$\text{since } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}v^2} dv = 1 \text{ (CDF of normal distribution)}$$

STOCHASTIC CALCULUS: DIFFERENTIATION OF FUNCTION OF A STOCHASTIC VARIABLE

- Ito's Lemma: Let $G(\xi, t)$ be continuous & at least twice differentiable function of a stochastic variable ξ and time t and let ξ be defined as the stochastic (Ito) process: $d\xi = a(\xi, t)dt + b(\xi, t)dW$
- Then, we have:
$$dG = \left(a \cdot \frac{\partial G}{\partial \xi} + \frac{\partial G}{\partial t} + \frac{1}{2} b^2 \cdot \frac{\partial^2 G}{\partial \xi^2} \right) dt + b \cdot \frac{\partial G}{\partial \xi} dW_t$$



Consider a continuous and differentiable function $G(\xi, t)$ of a stochastic variable ξ and t where ξ satisfies

$d\xi = a(\xi, t)dt + b(\xi, t)dW$. Taylor expansion of $G(\xi, t)$ is

$$dG = \frac{\partial G}{\partial \xi} d\xi + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial \xi^2} d\xi^2 + \frac{\partial^2 G}{\partial \xi \partial t} d\xi dt + \frac{1}{2} \frac{\partial^2 G}{\partial t^2} dt^2 + \dots$$

$$= \frac{\partial G}{\partial \xi} [a(\xi, t)dt + b(\xi, t)dW] + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial \xi^2} \left[a(\xi, t)dt + b(\xi, t)dW \right]^2$$

$$+ \frac{\partial^2 G}{\partial \xi \partial t} [a(\xi, t)dt + b(\xi, t)dW] dt + \frac{1}{2} \frac{\partial^2 G}{\partial t^2} dt^2 + \dots$$



We have $d\xi^2 = [a(\xi, t)dt + b(\xi, t)dW]^2$

$$= [a(\xi, t)dt + b(\xi, t)\sqrt{dt}Z]^2 = b^2 dt Z^2 + \text{h.o terms in } dt$$

Now, $E(Z^2) = 1$ whence $E(b^2 dt Z^2) = b^2 dt$.

Also $\text{Var}(Z^2) = 2$ whence $\text{Var}(b^2 dt Z^2) = 2b^4 dt^2 = 0$

if higher order terms than dt are neglected.

Now the Variance of a stochastic variable in time dt is proportional to dt not dt^2 .

Since $\text{Var}(b^2 dt Z^2) = 2b^4 dt^2$ is proportional to dt^2 it is too small to have a stochastic component. Hence, in the limit that higher order terms in dt are neglected, $(b^2 dt Z^2)$ and hence, $d\xi^2$ may be considered non-stochastic with a value of $b^2 dt$.



Thus, we get the Ito Lemma as :

$$dG = \left(a \cdot \frac{\partial G}{\partial \xi} + \frac{\partial G}{\partial t} + \frac{1}{2} b^2 \cdot \frac{\partial^2 G}{\partial \xi^2} \right) dt + b \cdot \frac{\partial G}{\partial \xi} dW$$

DRIFT RATE AND VARIANCE RATE

From Ito's Lemma, function $G(\xi, t)$ of a stochastic variable ξ , satisfies the Ito equation

$$dG = \left(a \cdot \frac{\partial G}{\partial \xi} + \frac{\partial G}{\partial t} + \frac{1}{2} b^2 \cdot \frac{\partial^2 G}{\partial \xi^2} \right) dt + b \cdot \frac{\partial G}{\partial \xi} dW$$

From this eq. we see that in an infinitesimal time interval dt , the process G is stochastic with a drift

rate $\left(a \cdot \frac{\partial G}{\partial \xi} + \frac{\partial G}{\partial t} + \frac{1}{2} b^2 \cdot \frac{\partial^2 G}{\partial \xi^2} \right)$, **variance rate** $\left(b \cdot \frac{\partial G}{\partial \xi} \right)^2$



STOCK PRICE DISTRIBUTION OVER AN INFINITESIMAL TIME PERIOD

- Stock prices are assumed to follow a Markov process.
- The Markov property of stock prices is consistent with the weak form of market efficiency i.e. that the current market price encapsulates its entire past history.
- The current price moves only when the market receives any relevant new information.
- The past history of prices is irrelevant for future price prediction since it is captured in the current price.



- The expected percentage return required by investors over an infinitesimal time period dt from a stock is independent of the stock's price. If investors require a 15% per annum expected return when the stock price is 10, then, they will also require a 15% per annum expected return when it is 50.
- The variance of percentage change in stock price in time dt is independent of the stock price. An investor is just as uncertain of the percentage return when the stock price is 50 as when it is 10.



The expected percentage return required by investors from a stock in an infinitesimal time dt is constant and independent of the stock price i.e.

$$E(R) = E\left[\frac{1}{dt}\left(\frac{dS}{S_0}\right)\right] = \mu \text{ or } E\left(\frac{dS}{S_0}\right) = \mu dt$$

$$\frac{dS}{S_0} = \mu dt + \text{randomness (with zero mean)}$$

The variability of the percentage change in stock price in a short period of time dt is the same irrespective of the stock price so that

$$\text{Var}\left(\frac{dS}{S_0}\right) = \sigma^2 dt \quad \text{or} \quad \text{SD}\left(\frac{dS}{S_0}\right) = \sigma \sqrt{dt}$$

$$\frac{dS}{S_0} = \mu dt + \sigma dW = \mu dt + \sigma \sqrt{dt} Z$$

$$\frac{dS}{S_0} = \mu dt + \sigma dW = \mu dt + \sigma \sqrt{dt} Z$$

$$dS = \mu S_0 dt + \sigma S_0 dW = \mu S_0 dt + \sigma S_0 \sqrt{dt} Z$$

$$S_{dt} - S_0 = dS \xrightarrow{\text{distribution}} N(\mu S_0 dt, \sigma^2 S_0^2 dt)$$

$$S_{dt} \xrightarrow{\text{distribution}} N(S_0 + \mu S_0 dt, \sigma^2 S_0^2 dt)$$